

MONODROMY OF FERMAT SURFACES AND MODULAR SYMBOLS FOR FERMAT CURVES

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ABSTRACT. We compute the modular symbols and the set of cuspidal modular symbols for the Fermat curve $x^n + y^n = z^n$. We use these computations to compute the monodromy of a certain fibration where the fibers are Fermat curves.

1. INTRODUCTION

This paper is motivated by a question raised by Eric Friedlander, which asks whether one can explicitly compute the étale cohomology groups of a variety using Artin good neighborhoods.

An Artin good neighborhood [Mil80] is a successive extension of affine curves and one can compute the group cohomology of its fundamental group using Serre-Hochschild spectral sequence [Ser94], [Sch14]. The first step in computing the spectral sequence is computing the monodromy of these extensions. As an example, we started by computing the cohomology of good neighborhoods of the Fermat surface $X : x^n + y^n + z^n = w^n$. It is still unknown to us whether one can combine the cohomological data of Artin neighborhoods to obtain the cohomology of the variety in practice since the intersection of any two Artin good neighborhoods does not have to be an Artin good neighborhood.

In this short paper, we are interested in computing the monodromy of the rational map $f : X \rightarrow \mathbb{P}^1$ from the Fermat surface X to the projective line, which is given by $f : (x : y : z : w) \mapsto (z : w)$. A generic fiber of the map f is the Fermat curve $C(n) : x^n + y^n = z^n$ and we use the modular interpretation of Fermat curves to describe the monodromy action by computing a set of generators for the homology group $H_1(C(n), \mathbb{Z})$ in terms of cuspidal modular symbols. (We work over the field of complex numbers.)

The background on the modular description of Fermat curves and the modular symbols are given in Section 2. In Section 3, we compute the modular symbols for Fermat curves and the generators of the first integral homology group. After describing the fibration and the induced action on a generic fiber, we compute the monodromy on the first homology group $H_1(C(n), \mathbb{Z})$ in Section 5. The followings are the main results of the paper. The first two propositions are proved in Section 3 and the Theorem 1.1 is proved in the last section.

Proposition 1.1. The group of modular symbols for the Fermat group $\Phi(n)$ is free of rank $(n^2 + 1)$ and the generators are given by

$$\begin{aligned} &\{[A^k B^l \tau] : 1 \leq k \leq (n-1), 0 \leq l \leq (n-1)\} \\ &\{[A^{n-1} B^l] : 0 \leq l \leq (n-1)\} \end{aligned}$$

and

$$[B^{n-1} \tau].$$

Proposition 1.2. The generators of the first homology group $H^1(X(\Phi(n)), \mathbb{Z})$ of $X(\Phi(n))$ are

$$\gamma_{i,j} := [A^i \tau] - [A^{i+1} B^{n-1} \tau] + [A^{i+1} B^j \tau] - [A^i B^{j+1} \tau]$$

for $1 \leq i \leq (n-2)$ and $0 \leq j \leq (n-2)$.

Theorem 1.1. Let σ_k be the generators of $\pi_1(V, y_0)$ as given in Section 4. Then the action of σ_k (for any k) on the homology group $H_1(C(n), \mathbb{Z})$ is given by the following:

$$\gamma_{i,j} \mapsto \gamma_{i+1,j} \text{ for } 1 \leq i \leq n-3, \quad 0 \leq j \leq n-2.$$

and if $i = n-2$, then

$$\gamma_{n-2,0} \mapsto \sum_{k=1}^{n-2} \gamma_{k,n-1-k}$$

and

$$\gamma_{n-2,j} \mapsto \sum_{l=0}^{j-1} \left(\sum_{s=j-l}^{n-l-2} \gamma_{s,l} - \gamma_{s,n-2-s+j-l} \right) - \sum_{i=j+1}^{n-2} \gamma_{i,n-i+j-1}$$

for $1 \leq j \leq n-2$.

2. BACKGROUND

2.1. Modular Interpretation of Fermat curves. Let $\bar{\mathbb{H}}$ denote the extended upper half plane. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\bar{\mathbb{H}}$ by fractional linear transformations. The quotient $G \backslash \bar{\mathbb{H}}$ of $\bar{\mathbb{H}}$ by the action of a finite index subgroup G of $\mathrm{SL}_2(\mathbb{Z})$ is a projective curve and it is denoted as $X(G)$. We will use the fact that the Fermat curve $x^n + y^n = z^n$ can be also described as $X(\Phi(n))$ for a subgroup $\Phi(n)$ of $\mathrm{SL}_2(\mathbb{Z})$.

We begin with defining a principle congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

$$\Gamma(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \pmod{2} \text{ and } b, c \equiv 0 \pmod{2} \right\}$$

It is known that $\Gamma(2) = \pm \langle A, B \rangle$ with

$$A := \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Let $\Phi(n) = \langle A^n, B^n, \Gamma(2)' \rangle$ be the subgroup of $\Gamma(2)$ generated by A^n, B^n , and the commutator subgroup $\Gamma(2)'$ of $\Gamma(2)$. Then $\bar{\mathbb{H}} \rightarrow X(\Gamma(2))$ induces a map

$$X(\Phi(n)) \longrightarrow X(\Gamma(2)).$$

It is known that $X(\Phi(n))$ is isomorphic to the Fermat curve $C(n) : x^n + y^n = z^n$ and the modular curve $X(\Gamma(2))$ is a genus 0 curve [Roh77], [Lon08]. Moreover, the map $X(\Phi(n)) \longrightarrow X(\Gamma(2))$ corresponds to the projection map

$$g : C(n) \longrightarrow \mathbb{P}^1 \\ (x : y : z) \mapsto (x^n : z^n)$$

The morphism g has degree n^2 with ramification at the points $0, 1$ and ∞ which are cusps of $X(\Gamma(2))$. At each of these points, the ramification degree is n and hence we can conclude that the modular curve $X(\Phi(n))$ has $3n$ cusps and there are exactly n of them lying above each of the points $0, 1$, and ∞ . Thus, the cuspidal points on $C(n)$ are the points $[x : y : z]$ with $xyz = 0$.

2.2. Cusps of $X(\Phi(n))$. Let $a_i = (0 : \zeta^i : 1)$, $b_i := (\zeta^i : 0 : 1)$ and $c_i := (\epsilon \zeta^i : 1 : 0)$ where $\epsilon = e^{\pi/n}$ and ζ is a primitive n 'th root of unity. Then using the map g , it is not hard to see that the points a_i correspond to the cusps above 0 , the points b_i correspond to 1 , and the points c_i represent the cusps above ∞ .

Let π denote the quotient map $\bar{\mathbb{H}} \rightarrow \Phi(n) \backslash \bar{\mathbb{H}}$. Then, $\pi(A.1) = \pi(B.1)$ since $BA^{-1}.1 = 1$ and $\pi(BA^{-1}.x) = \pi(A^{-1}Bx)$. Moreover, it is true that $\pi(A^k.1) = \pi(B^k.1)$. This can be proven by induction. Assume $\pi(A^{k-1}.1) = \pi(B^{k-1}.1)$, then $\pi(A^k.1) = \pi(A^{k-1}.B(1)) = \pi(BA^{k-1}.1) = \pi(B^k.1)$. (We use the fact that the quotient of $\Gamma(2)$ by $\Phi(n)$ is an abelian group.)

2.3. Automorphisms of $X(\Phi(n))$. In [Tze95], it is proven that the automorphism group of the Fermat curve $x^n + y^n = z^n$ is generated by two automorphisms of order n which are given by $(x : y : z) \mapsto (\zeta x : y : z)$, $(x : y : z) \mapsto (x : \zeta y : z)$ and the permutation group S_3 on three letters. On the other side, the normalizer of $\Phi(n)$ in $\mathrm{SL}_2(\mathbb{Z})$ defines a subgroup of the automorphism group of $X(\Phi(n))$. It is clear that A is in the normalizer of $\Phi(n)$ and note that $A \cdot \infty = \infty$. The only automorphism of order n fixing c_k for some k is given by $\delta : (x : y : z) \mapsto (\zeta x : \zeta y : z)$. Hence, the automorphism δ corresponds to some power of A .

2.4. Modular Symbols. Let \mathbb{M}_2 be the free abelian group with basis the set of symbols $\{\alpha, \beta\}$ with $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$ modulo the 3-term relations

$$\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\} = 0$$

and modulo any torsion. Define a left action of $\mathrm{SL}_2(\mathbb{Z})$ on \mathbb{M}_2 by letting $g \in \mathrm{SL}_2(\mathbb{Z})$ act by

$$g\{\alpha, \beta\} = \{g\alpha, g\beta\}$$

where the action of $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ on α and β is as it is defined in 2.1. Let G be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Then the subspace $\mathbb{M}_2(G)$ of modular symbols for G is the quotient of \mathbb{M}_2 by the submodule generated by the elements of the form $x - g.x$, for $x \in \mathbb{M}_2$ and $g \in G$, and modulo any torsion.

Suppose $g \in \mathrm{SL}_2(\mathbb{Z})$. Then the Manin symbol associated to g is $g.\{0, \infty\}$. Notice here that if $Gg = Gg'$, then $g.\{0, \infty\} = g'.\{0, \infty\}$ in $\mathbb{M}_2(G)$. Hence, there is a well-defined Manin symbol associated to each right coset of G in $\mathrm{SL}_2(\mathbb{Z})$. We will use the notation $[x]$ for the Manin symbol of Gx . (For more details, one can read [Ste07].)

Define a right action of G on the set of Manin symbols as follows:

$$\{0, \infty\} \cdot (hg) = h(g.\{0, \infty\}).$$

In other words, the group $\mathrm{SL}_2(\mathbb{Z})$ acts on Manin symbols as

$$h.[g] := [gh] \text{ for } g, h \in \mathrm{SL}_2(\mathbb{Z}).$$

Manin [Man72] showed that every modular symbol can be written as a \mathbb{Z} -linear combination of Manin symbols and determined the relations between these generators. Let

$$\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \tau = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad J = \begin{bmatrix} -1 & -0 \\ 0 & -1 \end{bmatrix}.$$

Theorem 2.1. [Man72] If x is a Manin symbol, then

- (1) $x + x\sigma = 0$
- (2) $x + x\tau + x\tau^2 = 0$
- (3) $x - xJ = 0$.

Moreover, these are all the relations between Manin symbols.

2.5. Homology of the curve $X(G)$. Let G be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Theorem 2.2. [Man72] The representation of any class $h \in H_1(X(G), \mathbb{Z})$ as a sum $\sum_{n_k} \{\alpha_k, \beta_k\}$ of Manin symbols can be chosen so that $\sum_k n_k(\phi(\beta_k) - \phi(\alpha_k)) = 0$ as a zero dimensional cycle on $X(G)$.

3. MODULAR SYMBOLS FOR FERMAT CURVES

We would like to compute $\mathbb{M}_2(\Phi(n))$. Hence, we will begin with finding coset representatives of $\Phi(n)$ in $\mathrm{SL}_2(\mathbb{Z})$. We already mentioned that the cosets of $\Phi(n)$ in $\Gamma(2)$ are given by $A^i B^j : 0 \leq i, j \leq (n-1)$.

Lemma 3.1. The set of matrices given below represents a complete set of coset representatives of $\Gamma(2)$ in $\mathrm{SL}_2(\mathbb{Z})$.

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Proof. Label the matrices with the given order in Lemma 3.1 as α_i 's where i runs through $1, \dots, 6$. Then it is enough to check $\alpha_i \alpha_j^{-1} \notin \Gamma(2)$ for any $i \neq j$. Since the index of $\Gamma(2)$ in $\mathrm{SL}_2(\mathbb{Z})$ is 6, we are done. \square

Notice that $\tau = \alpha_2$ and $\sigma = \alpha_3$. By Lemma 3.1, the right cosets of $\Phi(n)$ in $\mathrm{SL}_2(\mathbb{Z})$ are given by

$$A^i B^j \alpha_k, \quad 0 \leq i, j \leq (n-1), \quad 1 \leq k \leq 6.$$

In the following computations, the powers of A and B should be considered in \mathbb{Z}_n . We will begin with computing the relation 1 in Theorem 2.1.

3.1. σ -relations.

$$\begin{aligned} [\alpha_1 \sigma] &= [A^{-1} J \alpha_4], \quad [\alpha_2 \sigma] = [\alpha_5], \quad [\alpha_3 \sigma] = [\alpha_6 J] \\ [\alpha_4 \sigma] &= [A \alpha_1], \quad [\alpha_5 \sigma] = [\alpha_2], \quad [\alpha_6 \sigma] = [\sigma]. \end{aligned}$$

Hence by Equation 1, we have the following equalities:

$$\begin{aligned} (4) \quad & [A^k B^l \alpha_1] + [A^{k-1} B^l \alpha_4] = 0 \\ (5) \quad & [A^k B^l \alpha_2] + [A^k B^l \alpha_5] = 0 \\ (6) \quad & [A^k B^l \alpha_3] + [A^k B^l] = 0. \end{aligned}$$

Hence, $\mathbb{M}_2(\Phi(n))$ is generated by $[A^k B^l \alpha_j]$ for $0 \leq k, l \leq (n-1)$ and $j = 1, 2, 3$.

3.2. τ -relations.

$$\begin{aligned} [\alpha_1 \tau] &= [A^{-1}], \quad [\alpha_2 \tau] = [A \alpha_1], \quad [\alpha_3 \tau] = [A^{-1} \alpha_4] \\ [\alpha_4 \tau] &= [A B^{-1} \alpha_5], \quad [\alpha_5 \tau] = [B \alpha_3], \quad [\alpha_6 \tau] = [\tau] \end{aligned}$$

Hence by Equation 2,

$$\begin{aligned} (7) \quad & [A^k B^l] + [A^k B^l \alpha_2] + [A^{k+1} B^l \alpha_1] = 0 \\ (8) \quad & [A^k B^l \alpha_3] + [A^{k-1} B^l \alpha_4] + [A^k B^{l-1} \alpha_5] = 0. \end{aligned}$$

Using σ -relations, Equation 8 becomes

$$(9) \quad [A^k B^l] + [A^k B^l \alpha_1] + [A^k B^{l-1} \alpha_2] = 0$$

Hence by Equation 7, we conclude that the Manin symbols of the form $[A^k B^l]$ and $[A^k B^l \alpha_2]$ for $0 \leq k, l \leq (n-1)$ generate $\mathbb{M}_2(\Phi(n))$. Hence, using Equation 9 and Equation 7, we obtain the following relation:

$$(10) \quad [A^{k+1} B^l] + [A^{k+1} B^{l-1} \tau] = [A^k B^l] + [A^k B^l \tau]$$

for $0 \leq k \leq (n-1)$ and $0 \leq l \leq n-1$. Remember that $\alpha_2 = \tau$.

3.3. Proof of Proposition 1.1. We will use $x_{i,j} := [A^i B^j]$ and $y_{i,j} := [A^i B^j \tau]$ to ease the notation. We order the set $\{x_{i,j}, y_{i,j} : i = 1, \dots, n, j = 0, \dots, n-1\}$ as follows:

$$x_{i,j} \leq y_{k,l}$$

for any i, j, k, l and

$$x_{i,j} \leq x_{k,l} \text{ or } y_{i,j} \leq y_{k,l} \text{ if and only if } i < k \text{ or } i = k \text{ and } j \leq l.$$

Also note that i, j 's in the index set are defined up to congruence mod n . Hence, we see that if $i = n-1, j = 0$, then $x_{i+1,j-1} = x_{0,n-1}$. We may write the Equation 10 in a matrix form with respect to the ordered basis $\{x_{i,j}, y_{i,j} : i = 1, \dots, n-1, j = 0, \dots, n-1\}$ and use elementary row operations to find the solutions to (10). Let $R_{i,j}$ denote the ij 'th row of the resulting matrix, i.e. the row corresponding to the equation:

$$x_{i,j} + y_{i,j} = x_{i+1,j} + y_{i+1,j-1}$$

Notice that for $i \leq n-2$, first non-zero entry of each $R_{i,j}$ is 1 and it is the coefficient of $x_{i,j}$. Assume $i = n-1$. For j fixed, we add $\sum_{i \neq (n-1)} R_{i,j}$ to $R_{n-1,j}$ and denote the new $R_{n-1,j}$ by $R'_{n-1,j}$. Notice that the first n^2 entry of $R'_{n-1,j}$ is 0. Simply because

$$\sum_{i=0}^{n-1} (x_{i,j} - x_{i+1,j}) = 0$$

Similarly, when $j \geq 1$, the first non-zero term of $R'_{n-1,j}$ is the coefficient of $y_{0,j-1}$ and when $j = 0$, it is the coefficient of $y_{0,0}$ as it can be seen from the equation below.

$$\sum_{i=0}^{n-1} (y_{i,j} - y_{i+1,j-1}) = 0$$

Now, let $R''_{n-1,j} = \sum_{k \leq j} R'_{n-1,k}$ and replace $R'_{n-1,j}$ by $R''_{n-1,j}$.

Assume $j \neq n-1$. Then the first non-zero entry of $R''_{n-1,j}$ is 1 and it is the coefficient of $y_{0,j}$. If $j = n-1$, then $R''_{n-1,n-1}$ is the zero row since

$$\sum_k \left(\sum_i y_{i,k} - y_{i+1,k-1} \right) = 0$$

when $j \neq n-1$, we obtain the following equation:

$$(11) \quad \sum_{k \leq j} \left(\sum_{i=0}^{n-1} (y_{i,k} - y_{i+1,k-1}) \right) = \sum_{i=0}^{n-1} y_{i,n-1} - y_{i,j} = 0.$$

Note that (11) is not necessary for the proof of the Proposition but it will be useful in Section 5.

3.4. Proof of Proposition 1.2. We first compute that

$$\tau.\{0, \infty\} = \{1, 0\}.$$

Now,

$$\begin{aligned} A^i.\{1, 0\} - A^i B^j.\{1, 0\} &= \{A^i.1, A^i B^j.1\} \\ A^{i+1} B^{j-1}.\{1, 0\} - A^{i+1} B^{n-1}.\{1, 0\} &= \{A^{i+1} B^{j-1}.1, A^{i+1} B^{n-1}.1\}. \end{aligned}$$

By section 2.2, we know that $\phi(A^i(1)) = \phi(B^i(1))$. Hence,

$$\begin{aligned} \phi(A^{i+1} B^{n-1}(1)) &= \phi(A^{i+1} g A^{n-1}(1)) \text{ for some } g \in \Phi(n). \\ &= \phi(g' A^{i+1} A^{n-1}(1)) \text{ for some } g' \in \Phi(n). \\ &= \phi(A^i(1)). \end{aligned}$$

Similarly, one can show that

$$\phi(A^{i+1}B^{j-1}(1)) = \phi(A^iB^j(1)),$$

and we obtain that

$$\phi(A^i(1)) - \phi(A^iB^j(1)) + \phi(A^{i+1}B^{j-1}(1)) - \phi(A^{i+1}B^{n-1}(1)) = 0.$$

Hence by Theorem 2.2,

$$[A^i\tau] - [A^{i+1}B^{n-1}\tau] + [A^{i+1}B^{j-1}\tau] - [A^iB^j\tau]$$

is in $H^1(X(\Phi(n)), \mathbb{Z})$ for every i, j such that $1 \leq i \leq n-2$ and $0 \leq j \leq n-2$.

We also need to show that these generators are \mathbb{Z} -linearly independent. To ease the notation, we will use $y_{i,j}$ for $[A^iB^j\tau]$ as in the proof of Proposition 1.1.

Let $\gamma_{i,j}$ be the cycle $y_{i,0} - y_{i+1,n-1} + y_{i+1,j} - y_{i,j+1}$ and suppose that

$$(12) \quad \sum_{i,j} n_{i,j} \gamma_{i,j} = 0.$$

We will prove by induction that $n_{i,j} = 0$ for all $i = 1, \dots, n-2$ and for all $j = 0, \dots, n-2$.

We know by Proposition 1.1 that $y_{i,j}$'s are \mathbb{Z} -linearly independent. The coefficient of $y_{1,j+1}$ in the sum 12 is $-n_{1,j}$. Hence $n_{1,j} = 0$ for all $j \geq 1$. Similarly, for $j = 0$, the coefficient of $y_{1,0}$ is $\sum_j n_{1,j}$, hence $n_{1,0} = 0$. This proves the base step of our induction.

Assume $n_{i-1,j} = 0$ for all $j = 0, \dots, n-2$. Then, the coefficient of $y_{i,j}$ is $n_{i,j-1} + n_{i-1,j}$ and hence $n_{i,j-1} = 0$ for all $j = 1, \dots, n-2$ by the induction hypothesis. For $j = 0$, we look at the coefficient of $y_{i,0}$ which equals to $\sum_j (n_{i,j} + n_{i-1,0})$. Using what we just have proved, we see that $n_{i,0}$ must also equal to 0.

Hence $\{\gamma_{i,j} : 1 \leq i \leq n-2, 0 \leq j \leq n-2\}$ form a basis for the homology group.

Remark. Note that one can check that there are $(n-1)(n-2)$ generators as expected since

$$g(X(\Phi(n))) = g(C(n)) = (n-1)(n-2)/2.$$

4. FERMAT SURFACES

4.1. Definition of Monodromy. Let X and Y be algebraic varieties over \mathbb{C} . Suppose that $f : X \rightarrow Y$ is a fibration. Let y_0 be in Y and let X_0 be the fiber of f at y_0 .

For γ in $\pi_1(Y, y_0)$, define $H : X_0 \times [0, 1] \rightarrow Y$ as $H(x, t) := \gamma(t)$ for all $x \in X_0$. Then by the homotopy lifting property of f , there exists a lift $\tilde{H} : X_0 \times [0, 1] \rightarrow X$ as in the following diagram [DK01].

$$\begin{array}{ccc} X_0 \times \{0\} & \xrightarrow{\quad} & X \\ \downarrow & \nearrow \tilde{H} & \downarrow f \\ X_0 \times [0, 1] & \xrightarrow{H} & Y \end{array}$$

The commutativity of the above diagram induces a morphism $X_0 \rightarrow X_0$ (at $t = 1$) which induces a homomorphism on the homology groups and we obtain an action of the fundamental group $\pi_1(Y, y_0)$ of Y on $H_1(X_0, \mathbb{Z})$:

$$\rho : \pi_1(Y, y_0) \longrightarrow \text{Aut}(H_1(X_0, \mathbb{Z}))$$

which we call the monodromy action. It is worth to note here that the map $X_0 \rightarrow X_0$ induced by the diagram above is a self homotopy equivalence and unique up to a homotopy.

4.2. Fermat Surfaces. Let X denote the surface given by the equation

$$x^n + y^n + z^n = w^n \subset \mathbb{P}_{\mathbb{C}}^3.$$

Consider the rational map $f : X \rightarrow \mathbb{P}^1$ given by

$$f : (x : y : z : w) \mapsto (z : w).$$

Blowing up the variety X successively at the points in the set $B = \{(x : y : 0 : 0) \in X\}$, we obtain a variety \bar{X} , $\bar{f} : \bar{X} \rightarrow \mathbb{P}^1$ and $\varphi : \bar{X} \rightarrow X$ such that \bar{X} and X are isomorphic outside the set B . A generic fiber of f is given by the equation $x^n + y^n = 1$. The morphism \bar{f} is proper, hence the fibers are projective curves and a generic fiber of \bar{f} is isomorphic to the curve

$$x^n + y^n = z^n.$$

and singular fibers are given by $x^n + y^n = 0$ when $z^n = w^n$. Let U denote the variety $\bar{f}^{-1}(\mathbb{P}^1 - S)$ with $S = \{[\zeta_n^k : 1] \in \mathbb{P}^1 : k = 0, \dots, n-1\}$. We obtain a fibration $U \rightarrow V$ where V denotes $\mathbb{P}^1 - S$ by Ehreman's theorem [Ehr95].

let y_0 be the point $[0 : 1] \in \mathbb{P}^1$. The generators of the fundamental group $\pi_1(V, y_0)$ can be formulated in the following way:

If n is even, then take leaves of the polar rose $r = 2\cos(\frac{n}{2}\theta)$ and take leaves of $r = 2\cos(n\theta)$ otherwise (in the $w = 1$ complex plane). When $n = 3$, one can visualize these generators as in the following figure.

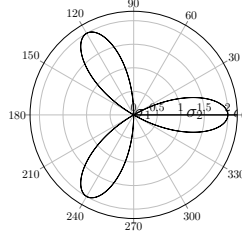


FIGURE 1. Generators of $\pi_1(V, y_0)$ in $w = 1$ plane.

let σ be the loop around the singular point $[\zeta_n^k : 1]$. Remember that ρ denotes the monodromy representation. Note that $1 - \sigma(t)^n$ is a loop and hence we can find continuous functions $r(t), g(t)$ such that $1 - \sigma(t)^n = r(t)e^{2\pi i g(t)}$. We may assume that $g(0) = 0$ and $g(1) = 1$. Let $(u : v : w)$ be a point on the fiber $x^n + y^n = z^n$. Note that $\sigma(t)$ lies in the complex plane $w = 1$. Hence, locally we may assume that U is given by the equation $x^n + y^n = w^n(1 - u^n)$ for $u^n \neq 1$. Hence, the morphism f corresponds to the morphism $(x : y : w : u) \mapsto u$. Remember that we pick the base point as $u = 0$ (y_0). Let $(x : y : w : 0)$ be a point in the fiber of $u = 0$. Then define

$$\tilde{\sigma} : I \rightarrow U$$

$$\tilde{\sigma}(t) = (\lambda(t)x, \lambda(t)y : w : \sigma(t))$$

where $\lambda(t) = r(t)^{1/n} e^{\frac{2\pi i g(t)}{n}}$. One can check that $w^n(1 - \sigma(t)^n) = (1 - \sigma(t)^n)(x^n + y^n)$. Also notice that $\tilde{\sigma}(0) = (x : y : w)$ and $\tilde{\sigma}(1) = (\zeta_n x : \zeta_n y : w)$. Hence, this lift $\tilde{\sigma}$ induces an action on the fiber $x^n + y^n = w^n$ as

$$(x : y : w) \mapsto (\zeta_n x, \zeta_n y : w).$$

In particular, this action is independent of the choice of σ_k .

5. CALCULATION OF THE MONODROMY

In Section 4, we showed that the action of each generator of the fundamental group on a generic fiber is identical. Using our conclusion from 2.3, it is enough to compute the action of a power of A on the modular symbols $\gamma_{i,j}$ that we found in Proposition 1.2 to compute the monodromy of the fibration described in section 4.

Remember that $C(n)$ denotes the Fermat curve $x^n + y^n = z^n$.

5.1. Proof of Theorem 1.1. We recall from equation (11) that

$$\sum_{i=0}^{n-1} (y_{i,n-1} - y_{i,j}) = 0$$

for any $0 \leq j \leq n-2$. If $i < n-2$, then it is clear that $A.\gamma_{i,j} = \gamma_{i+1,j}$. Assume $i = n-2$, then

$$\begin{aligned} A.\gamma_{n-2,j} &= A.(y_{n-2,0} - y_{n-1,n-1} + y_{n-1,j} - y_{n-2,j+1}) \\ &= y_{n-1,0} - y_{0,n-1} + y_{0,j} - y_{n-1,j+1} \\ &= y_{n-1,0} - y_{0,n-1} + y_{0,j} - y_{n-1,j+1} + \sum_{i=0}^{n-1} (y_{i,n-1} - y_{i,j}). \end{aligned}$$

We will compute the answer in two separate cases $j = 0$ and $j \geq 1$. Assume $j = 0$, then

$$\begin{aligned} A.\gamma_{n-2,0} &= y_{n-1,0} - y_{0,n-1} + y_{0,0} - y_{n-1,1} + \sum_{i=0}^{n-1} (y_{i,n-1} - y_{i,0}) \\ &= y_{n-1,0} - y_{n-1,1} + \sum_{i=1}^{n-1} (y_{i,n-1} - y_{i,0}) \\ &= -y_{n-1,1} + y_{1,n-1} + \sum_{i=2}^{n-1} y_{i,n-1} - \sum_{i=1}^{n-2} y_{i,0} \\ &= \sum_{i=1}^{n-2} (y_{i,n-i} - y_{i+1,n-i-1}) + \sum_{i=2}^{n-1} (y_{i,n-1} - y_{i-1,0}) \\ &= \sum_{i=1}^{n-2} (y_{i+1,n-1} - y_{i,0} + y_{i,n-i} - y_{i+1,n-i-1}) \\ &= -\sum_{i=1}^{n-2} \gamma_{i,n-i-1}. \end{aligned}$$

Assume $j \geq 1$. The proof is similar to the case $j = 0$ but it requires more work.

$$\begin{aligned}
A.\gamma_{n-2,j} &= y_{n-1,0} - y_{n-1,j+1} + \sum_{i=1}^{n-1} (y_{i,n-1} - y_{i,j}) \\
&= y_{n-1,0} - y_{n-1-j,j} + y_{j+1,n-1} - y_{n-1,j+1} + \sum_{\substack{i=1 \\ i \neq j+1}}^{n-1} (y_{i,n-1} - y_{i-1-j,j}) \quad (\text{replace } i+j+1 \text{ by } i) \\
&= y_{n-1,0} - y_{n-1-j,j} + \left(\sum_{i=j+2}^{n-1} y_{i,n-1} - y_{i-1-j,j} \right) + \left(\sum_{i=1}^j y_{i,n-1} - y_{n+i-1-j,j} \right) \\
&\quad + (y_{j+1,n-1} - y_{n-1,j+1}).
\end{aligned}$$

We first compute that

$$\sum_{i=j+2}^{n-1} (y_{i,n-1} - y_{i-1-j,j}) = \sum_{i=j+2}^{n-1} (y_{i,n-1} - y_{i-1,0}) + \sum_{i=j+2}^{n-1} (y_{i-1,0} - y_{i-1-j,j}).$$

We can easily calculate that

$$\begin{aligned}
&\left(\sum_{i=j+2}^{n-1} y_{i,n-1} - y_{i-1,0} \right) + (y_{j+1,n-1} - y_{n-1,j+1}) \\
&= \sum_{i=j+2}^{n-1} (y_{i,n-1} - y_{i-1,0}) + \sum_{i=j+1}^{n-2} (y_{i,n+j-i} - y_{i+1,n+j-i-1}) \\
&= \sum_{i=j+1}^{n-2} (y_{i+1,n-1} - y_{i,0} + y_{i,n+j-i} - y_{i+1,n+j-i-1}) \\
&= - \sum_{i=j+1}^{n-2} \gamma_{i,n-i+j-1}.
\end{aligned}$$

Now, we are left with showing that

$$(13) \quad \left(\sum_{i=j+2}^{n-1} y_{i-1,0} - y_{i-1-j,j} \right) + y_{n-1,0} - y_{n-1-j,j} + \left(\sum_{i=1}^j y_{i,n-1} - y_{n+i-1-j,j} \right)$$

$$(14) \quad = \sum_{l=0}^{j-1} \left(\sum_{s=j-l}^{n-l-2} \gamma_{s,l} - \gamma_{s,n-2-s+j-l} \right).$$

On the left hand side of equation (13),

$$\left(\sum_{i=j+2}^{n-1} y_{i-1,0} - y_{i-1-j,j} \right) + y_{n-1,0} - y_{n-1-j,j} = \left(\sum_{i=j+1}^{n-1} y_{i,0} - y_{i-j,j} \right).$$

Hence, we are left with the sum

$$\left(\sum_{i=j+1}^{n-1} y_{i,0} - y_{i-j,j} \right) + \left(\sum_{i=1}^j y_{i,n-1} - y_{n+i-1-j,j} \right).$$

We can rearrange the first sum as the following:

$$\begin{aligned}
\sum_{i=j+1}^{n-1} y_{i,0} - y_{i-j,j} &= \sum_{i=j+1}^{n-1} \left(\sum_{l=i-j+1}^i y_{l,i-l} - y_{l-1,i-l+1} \right) \\
&= \sum_{l=0}^{j-1} \left(\sum_{k=0}^{n-2-j} y_{1+j+k-l,l} - y_{j+k-l,l+1} \right) \\
&= \sum_{l=0}^{j-1} \left(\sum_{s=j-l}^{n-l-2} y_{1+s,l} - y_{s,l+1} \right).
\end{aligned}$$

Similarly, we can rearrange the second sum as:

$$\begin{aligned}
\sum_{i=1}^j y_{i,n-1} - y_{n+i-1-j,j} &= \sum_{k=1}^j \left(\sum_{l=k}^{n-2+k-j} y_{l,n-1+k-l} - y_{l+1,n-2+k-l} \right) \\
&= \sum_{s=0}^{j-1} \left(\sum_{l=j-s}^{n-2-s} y_{l,n-1-l+j-s} - y_{l+1,n-2-l+j-s} \right).
\end{aligned}$$

Combining these two sums,

$$\begin{aligned}
&= \sum_{l=0}^{j-1} \left(\sum_{s=j-l}^{n-l-2} y_{1+s,l} - y_{s,l+1} \right) + \sum_{s=0}^{j-1} \left(\sum_{l=j-s}^{n-2-s} y_{l,n-1-l+j-s} - y_{l+1,n-2-l+j-s} \right) \\
&= \sum_{l=0}^{j-1} \left(\sum_{s=j-l}^{n-2-l} y_{1+s,l} - y_{s,l+1} + y_{s,n-1-l+j-s} - y_{s+1,n-2-l+j-s} \right) \\
&= \sum_{l=0}^{j-1} \left(\sum_{s=j-l}^{n-2-l} \gamma_{s,l} - \gamma_{s,n-2-s+j-l} \right).
\end{aligned}$$

Corollary 5.0.1. The action of A^d for $d \geq 1$ on the homology group $H_1(X(\Phi(n)), \mathbb{Z})$ is given by the following:

$$\gamma_{i,j} \mapsto \gamma_{i+d,j} \text{ for } 1 \leq i \leq n-2 \text{ and } i \neq n-1-d, 0 \leq j \leq n-2.$$

and if $i = n-1-k$, then

$$\gamma_{n-1-d,0} \mapsto \sum_{k=1}^{n-2} \gamma_{k,n-1-k}$$

and

$$\gamma_{n-1-d,j} \mapsto \sum_{l=0}^{j-1} \left(\sum_{s=j-l}^{n-l-2} \gamma_{s,l} - \gamma_{s,n-2-s+j-l} \right) - \sum_{k=1+j}^{n-2} \gamma_{k,n-2-k}$$

for $1 \leq j \leq n-2$.

Remark. Let $n = 3$. Then the monodromy of each generator is given by $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ which coincides with monodromy of an elliptic surface with a singularity of three lines intersecting at a single point given in [Kod64], [Kod66], [Mir89].

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REFERENCES

- [DK01] James F. Davis and Paul Kirk. *Lecture notes in algebraic topology*, volume 35 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.
- [Ehr95] Charles Ehresmann. Les connexions infinitésimales dans un espace fibré différentiable. In *Séminaire Bourbaki, Vol. 1*, pages Exp. No. 24, 153–168. Soc. Math. France, Paris, 1995.
- [Kod64] K. Kodaira. On the structure of compact complex analytic surfaces. I. *Amer. J. Math.*, 86:751–798, 1964.
- [Kod66] K. Kodaira. On the structure of compact complex analytic surfaces. II. *Amer. J. Math.*, 88:682–721, 1966.
- [Lon08] Ling Long. Finite index subgroups of the modular group and their modular forms. In *Modular forms and string duality*, volume 54 of *Fields Inst. Commun.*, pages 83–102. Amer. Math. Soc., Providence, RI, 2008.
- [Man72] Ju. I. Manin. Parabolic points and zeta functions of modular curves. *Izv. Akad. Nauk SSSR Ser. Mat.*, 36:19–66, 1972.
- [Mil80] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.
- [Mir89] Rick Miranda. *The basic theory of elliptic surfaces*. Dottorato di Ricerca in Matematica. [Doctorate in Mathematical Research]. ETS Editrice, Pisa, 1989.
- [Roh77] David E. Rohrlich. Points at infinity on the Fermat curves. *Invent. Math.*, 39(2):95–127, 1977.
- [Sch14] Leila Schneps. The Grothendieck-Serre correspondence. In *Alexandre Grothendieck: a mathematical portrait*, pages 193–230. Int. Press, Somerville, MA, 2014. Revised reprint of the 2001 original.
- [Ser94] Jean-Pierre Serre. *Cohomologie galoisienne*, volume 5 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, fifth edition, 1994.
- [Ste07] William Stein. *Modular forms, a computational approach*, volume 79 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007. With an appendix by Paul E. Gunnells.
- [Tze95] Pavlos Tzermias. The group of automorphisms of the Fermat curve. *J. Number Theory*, 53(1):173–178, 1995.

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